UNITARY EQUIVALENCE TO A COMPLEX SYMMETRIC MATRIX

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Abstract. We present a necessary and sufficient condition for a 3×3 matrix to be unitarily equivalent to a symmetric matrix with complex entries, and an algorithm whereby an arbitrary 3×3 matrix can be tested. This test generalizes to a necessary and sufficient condition that applies to almost every $n \times n$ matrix. The test is constructive in that it explicitly exhibits the unitary equivalence to a complex symmetric matrix.

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1. Introduction

A square matrix T with complex entries is called *complex symmetric* if it is symmetric across the main diagonal (i.e., $T = T^t$ where the superscript t denotes the transpose operation). The phrase *complex* symmetric emphasizes the distinction between these matrices and the real symmetric matrices, which are also self-adjoint. Complex symmetric matrices have been studied extensively ([10, Section 4.4], for example), and they have many applications to science and mathematics [1, 5, 11, 12].

In this thesis, we are primarily interested in the linear transformations on $\mathbb{C}^n$ that are induced by complex symmetric matrices. In particular, we will think of a given matrix as acting on the inner product space $(\mathbb{C}^n, \|\cdot\|_2)$, where $\|\cdot\|_2$ is the Euclidean norm. The question that we investigate is how to tell if a given linear transformation arises from a complex symmetric matrix.

By choosing a preferred basis, we can write our given linear transformation as a matrix. However, this choice of basis, and thus the resulting matrix, is not unique. It turns out that every matrix is similar to a complex symmetric matrix (Theorem 3.7), and thus if we are allowed to choose *any* basis for $\mathbb{C}^n$, then every linear transformation can be represented by a complex symmetric matrix.

Therefore, we would like to be more careful in selecting a basis. In an inner product space, the bases that respect the geometry of the space are the *orthonormal bases.* The natural question to ask is whether or not there is an orthonormal basis with respect to which our linear transformation is represented by a complex symmetric matrix. The matrices that can represent the same linear transformation with respect to different orthonormal bases are *unitarily equivalent.* We can rephrase our question as follows.

Question. Given a square matrix, can we tell if it is unitarily equivalent to a complex symmetric matrix?

It is not unusual to take a special class of matrices and consider the larger set of matrices that are equivalent to a member of this class via some equivalence relation. For example, we study

- Diagonalizable matrices (those that are similar to a diagonal matrix)

- Normal matrices (those that are unitarily equivalent to a diagonal matrix)
- Invertible matrices (those that are row equivalent to the identity)

Definition. A matrix that is unitarily equivalent to a complex symmetric matrix is called *UECSM*. We will similarly abbreviate "complex symmetric matrix" as *CSM*.

The study of UECSM's is also motivated by the recent study of complex symmetric operators [7, 8]. In finite dimensions, the complex symmetric operators correspond to the UECSM's, and so our question can be restated in terms of complex symmetric operators.

Question. Given a square matrix, can we tell if it induces a complex symmetric operator?

This problem is made more difficult by the aforementioned result that every matrix is similar to a complex symmetric matrix. Consequently, similarity invariants (such as trace, determinant and eigenvalues) are not very useful in answering the above question. For instance, the following three matrices are similar, but only one of them is UECSM:

$$T_1 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{bmatrix}.$$

For more, see Example 6.6. In this thesis, we present a complete answer to the above questions for spaces of dimension three and fewer, and a partial answer to the general case. We also provide an algorithm whereby one can test a given matrix for being UECSM. The algorithm applies to any 3×3 matrix, and almost every $n \times n$ matrix (with respect to the Lebesgue measure on $\mathbb{C}^{n^2}$). When a matrix is UECSM, the algorithm provides the unitary matrix and explicitly exhibits the unitary equivalence to a complex symmetric matrix. These results were first presented in [13].

2. Background

We begin with a discussion of complex Euclidean space $(\mathbb{C}^n, \|\cdot\|_2)$. In general, we will simply write $\|\cdot\|$ for $\|\cdot\|_2$. Recall that for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, this norm is given by

$$\|x\| = \sqrt{\sum_{k=1}^{n} |x_k|^2} = \sqrt{\langle x, x \rangle},$$

where $\langle \cdot, \cdot \rangle$ represents the standard inner product,

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

The inner product has the following properties:

- $\langle x, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for any $x, y, z \in \mathbb{C}^n$ and any $\alpha, \beta \in \mathbb{C}$.
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for any $x, y \in \mathbb{C}^n$.
- $\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$ for any $x, y, z \in \mathbb{C}^n$ and any $\alpha, \beta \in \mathbb{C}$.

These properties are not independent; the fourth can easily be derived from the second and third.

Definition. Two vectors $x, y \in \mathbb{C}^n$ are called *orthogonal* if $\langle x, y \rangle = 0$. If M is a subspace of $\mathbb{C}^n$, then the *orthogonal complement of* M is the subspace defined by

$$M^\perp = \{x \in \mathbb{C}^n : \langle x, y \rangle = 0 \text{ for all } y \in M\}.$$

Definition. A basis is called *orthogonal* if every pair of distinct basis vectors is orthogonal. A basis is called *orthonormal* if it is orthogonal and every basis element is a unit vector.

More concisely, a set $\{e_1, \dots, e_n\} \subset \mathbb{C}^n$ is an orthonormal basis if and only if

$$\langle e_i, e_j \rangle = \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

The symbol $\delta_{i,j}$ is called the *Kronecker delta*. A typical orthonormal basis is the standard basis $\{e_i\}$, where e_i is the vector that has a 1 in the ith position and 0's everywhere else.

If $\{e_1, \dots, e_n\}$ is an orthonormal basis, we get the *reconstruction formula*

$$x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i,$$

for all $x \in \mathbb{C}^n$.

Closely tied to orthonormal bases is the notion of a unitary matrix. However, we need a couple of definitions before we can continue.

Definition. The set of $n \times n$ matrices with complex entries will be denoted $M_n(\mathbb{C})$. If $T = (t_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$, then the matrix

$$T^* = (\overline{t_{ji}})_{i,j=1}^n$$

is called the *adjoint* of T. A matrix satisfying $T = T^*$ is called *self-adjoint*.

Proposition 2.1. *The adjoint operation has the following properties:*

(i) $(T^*)^* = T$,
(ii) $(T+S)^* = T^* + S^*$,
(iii) $(\alpha T)^* = \overline{\alpha} T^*$,
(iv) $(TS)^* = S^* T^*$,
(v) $\langle Tx, y \rangle = \langle x, T^* y \rangle$ *for every* $x, y \in \mathbb{C}^n$.

We will use the above properties frequently, especially (v).

With the notion of adjoint in hand, we are now ready to define what it means for a matrix to be unitary. There are many equivalent definitions of a unitary matrix. Consider the following proposition:

Proposition 2.2. *Let $U \in M_n(\mathbb{C})$. The following are equivalent:*

(i) $\|Ux\| = \|x\|$ *for all* $x \in \mathbb{C}^n$*,*
(ii) $\langle x, y\rangle = \langle Ux, Uy\rangle$ *for all* $x, y \in \mathbb{C}^n$*,*
(iii) $U^*U = I$*,*
(iv) $UU^* = I$*,*
(v) The rows of U *form an orthonormal basis,*
(vi) The columns of U *form an orthonormal basis,*
(vii) U *maps orthonormal bases to orthonormal bases.*

Definition. A matrix that satisfies any (and therefore all) of the hypotheses of Proposition 2.2 is called *unitary.*

The proof of Proposition 2.2 is an elementary exercise in linear algebra, and we will not discuss it here. However, we do state a useful formula that furnishes a proof of the equivalence of (i) and (ii).

Proposition 2.3 (*The polarization identity*)**.** *For any $x, y \in \mathbb{C}^n$, we have*

$$4\langle x, y\rangle = (\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2).$$

Proposition 2.2 says that unitary matrices preserve both the lengths of vectors and the "angles" between them. The actions of unitary matrices correspond to the rigid motions of $\mathbb{C}^n$ that fix the origin. The proposition also says that unitary matrices take orthonormal bases to orthonormal bases, and thus they can be thought of as a transition matrix from one orthonormal basis to another.

Definition. If $A, B \in M_n(\mathbb{C})$ and there is some unitary matrix U such that $A = U^*BU$, then A and B are said to be *unitarily equivalent.*

It is clear that unitary equivalence is an equivalence relation, and that if two matrices are unitarily equivalent then they are also similar.

Proposition 2.4. *Two matrices represent the same linear transformation with respect to different orthonormal bases if and only if the matrices are unitarily equivalent.*

This proposition says that a linear transformation T can be represented by a complex symmetric matrix with respect to some orthonormal basis if and only if an arbitrary matrix representing T (with respect to an orthonormal basis) is unitarily equivalent to a complex symmetric matrix (or UECSM).

We conclude this section by listing some basic properties of the set of UECSM's.

Proposition 2.5.

(i) *If T is UECSM, then so is $T + \alpha I$ for any $\alpha \in \mathbb{C}$.*
(ii) *If T is UECSM, then so is αT for any $\alpha \in \mathbb{C}$.*
(iii) *If T is UECSM, then so is T^*.*
(iv) *If T and S are UECSM, then so is $T \oplus S$.*
(v) *The set of all UECSM's is (topologically) closed.*

Proof. Everything but (v) is left to the reader. Suppose that $T_n \to T$ and that there exist unitaries U_n such that $U_n^* T_n U_n$ is a complex symmetric matrix. Since the set of unitary matrices is compact (i.e. closed and bounded), we can assume without loss of generality that $U_n \to U$ for some unitary matrix U by passing to a subsequence. Thus $U_n^* T_n U_n \to U^* T U$, and $U^* T U$ is complex symmetric since the set of complex symmetric matrices is closed. □

Remark. Even if T and S are UECSM, it does not generally hold that $T + S$ and TS are UECSM.

3. Conjugations

For the remainder of this thesis, we will use matrices and linear transformations interchangeably. Unless otherwise specified, the matrix of a linear transformation is taken with respect to the standard basis. For now, we will focus on the linear transformations more than the matrices.

Definition. A *conjugation* is a function $C : \mathbb{C}^n \to \mathbb{C}^n$ that has the following properties:

(i) $C(\alpha x + \beta y) = \overline{\alpha} Cx + \overline{\beta} Cy$ for all $x, y \in \mathbb{C}^n$ and all $\alpha, \beta \in \mathbb{C}$ (conjugate-linear),
(ii) $C(C(x)) = x$ for all $x \in \mathbb{C}^n$ (involutive),
(iii) $\|Cx\| = \|x\|$ for all $x \in \mathbb{C}^n$ (isometric).

Example 3.1. The simplest example of a conjugation on $\mathbb{C}^n$ is the map

$$C(x_1, \ldots, x_n) = (\overline{x_1}, \ldots, \overline{x_n}),$$

which is just coordinate-wise complex conjugation. This is called the *standard conjugation.* A related, but slightly more complicated, conjugation is the "flip conjugation",

$$C(x_1, \ldots, x_n) = (\overline{x_n}, \ldots, \overline{x_1}).$$

Example 3.2. A more general way to obtain a conjugation is the following. Let U be a unitary matrix, and C be any conjugation. If we define $J = UCU^*$, then it is immediate that J is also a conjugation. In the case where C is the standard conjugation, we can make sense of how this new conjugation behaves. The standard conjugation acts on x by performing complex conjugation on the coefficients of x with respect to the standard basis $\{e_i\}$. The new conjugation acts by performing complex conjugation on the coefficients of x with respect to the orthonormal basis $\{Ue_i\}$. We will see in Lemma 3.6 that all conjugations arise in this way.

We saw earlier that any linear isometry also preserves inner products. We now prove a similar result for conjugations.

Proposition 3.3. *If C is a conjugation, then $\langle x, y \rangle = \langle Cy, Cx \rangle$ for all $x, y \in \mathbb{C}^n$.*

Proof. The following calculation uses Proposition 2.3 (the polarization identity) and the fact that conjugations are both conjugate-linear and isometric.

$$\begin{aligned}
4\langle x, y\rangle &= (\|x+y\|^2 - \|x-y\|^2) + i(\|x+iy\|^2 - \|x-iy\|^2) \\
&= (\|C(x+y)\|^2 - \|C(x-y)\|^2) + i(\|C(x+iy)\|^2 - \|C(x-iy)\|^2) \\
&= (\|Cx+Cy\|^2 - \|Cx-Cy\|^2) + i(\|Cx-iCy\|^2 - \|Cx+iCy\|^2) \\
&= (\|Cx+Cy\|^2 - \|Cx-Cy\|^2) - i(\|Cx+iCy\|^2 - \|Cx-iCy\|^2) \\
&= 4\overline{\langle Cx, Cy\rangle} \\
&= 4\langle Cy, Cx\rangle.
\end{aligned}$$

□

Definition. We say that a linear transformation T is C-symmetric for some conjugation C if $T = CT^*C$.

Remark. The set of C-symmetric matrices form a linear subspace of $\mathcal{M}_n(\mathbb{C})$ for any fixed conjugation C. This set is also closed under taking adjoints (i.e., it is $*$-closed).

We will characterize the UECSM matrices in terms of C-symmetry, but first we discuss a motivating example.

Example 3.4. Let C be the standard conjugation, T be any matrix, and $\{e_i\}$ be the standard basis. By using the fact that $Ce_i = e_i$ and the reconstruction formula for orthonormal bases, we can calculate

$$\begin{aligned}
CT^*Ce_i &= CT^*e_i \\
&= C\sum_{j=1}^{n}\langle T^*e_i, e_j\rangle e_j \\
&= \sum_{j=1}^{n}\overline{\langle T^*e_i, e_j\rangle}Ce_j \\
&= \sum_{j=1}^{n}\langle Te_j, e_i\rangle e_j.
\end{aligned}$$

Recall that the (i,j)-th entry of the matrix T is given by $\langle Te_j, e_i\rangle$, and so $\langle Te_j, e_i\rangle = \langle T^t e_i, e_j\rangle$. Substituting yields

$$\begin{aligned}\sum_{j=1}^{n}\langle Te_j, e_i\rangle e_j &= \sum_{j=1}^{n}\langle T^t e_i, e_j\rangle e_j \\ &= T^t e_i.\end{aligned}$$

Since CT^*C and T^t are linear and agree on a basis, we have that $T^t = CT^*C$. We conclude that a matrix is C-symmetric (where here C is the standard conjugation) if and only if $T = T^t$.

The previous example generalizes to the following characterization of UECSM's in terms of C-symmetry [7, Proposition 2].

Lemma 3.5. *An $n \times n$ matrix T is unitarily equivalent to a complex symmetric matrix if and only if there exists a conjugation C for which the linear transformation induced by T is C-symmetric.*

We delay the proof of Lemma 3.5 until after we develop one more piece of machinery [7, Lemma 1].

Lemma 3.6. *If C is a conjugation on $\mathbb{C}^n$, then there is an orthonormal basis $\{e_i\}_{i=1}^n$ such that $Ce_i = e_i$ for all i.*

Proof. Let $V = \{Cx + x : x \in \mathbb{C}^n\}$, and observe that $Cy = y$ for every $y \in V$. By the definition of a conjugation, V is a real-linear subspace of $\mathbb{C}^n$ and thus it is a real inner product space with respect to the inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis for V with respect to this inner product. We already know that $Ce_i = e_i$ for every i, as each $e_i \in V$. We will now show that $\{e_i\}$ is a (complex) orthonormal basis for $\mathbb{C}^n$. Since $\{e_i\}$ is orthonormal with respect to the real inner product $\mathrm{Re}\langle\cdot,\cdot\rangle$, we have that

$$\mathrm{Re}\langle e_i, e_j\rangle = \delta_{i,j}.$$

However, since

$$\begin{aligned}\langle e_i, e_j\rangle &= \langle Ce_i, Ce_j\rangle \\ &= \langle e_j, e_i\rangle \\ &= \overline{\langle e_i, e_j\rangle},\end{aligned}$$

we have that $\langle e_i, e_j\rangle \in \mathbb{R}$ and thus $\{e_i\}$ is orthonormal with respect to the standard inner product on $\mathbb{C}^n$. Any orthogonal set will be linearly independent, but it remains to show that the complex span of $\{e_i\}$ is $\mathbb{C}^n$. Let $x \in \mathbb{C}^n$, and observe that

$$\begin{aligned}x &= \tfrac{1}{2}(Cx + x) + i\tfrac{1}{2i}(x - Cx) \\ &= \tfrac{1}{2}(Cx + x) + i\tfrac{1}{2}(-ix + C(-ix)).\end{aligned}$$

This shows that every element of $\mathbb{C}^n$ can be written as a complex linear combination of elements of V. Every element of V can be written as a real linear combination of the e_i, and thus every element of $\mathbb{C}^n$ can be written as a complex linear combination of the e_i. We conclude that $\{e_i\}$ spans $\mathbb{C}^n$ and thus satisfies the conclusion of the lemma. $\square$

Definition. Let C be a conjugation and let $\{e_i\}_{i=1}^n$ be as in Lemma 3.6. We refer to $\{e_i\}$ as a *C-real basis*. Note that C-real bases are assumed to be orthonormal, although sometimes we will say "C-real orthonormal basis" for emphasis.

Remark. A consequence of the previous lemma is that any conjugation C can be written in the form $C = UJU^*$, where J is the standard conjugation. Also note that a C-real basis will never be unique. In fact, the proof of Lemma 3.6 shows how to construct infinitely many C-real bases for any conjugation.

We are now ready to prove Lemma 3.5.

Proof of Lemma 3.5. Let $T \in M_n(\mathbb{C})$ and suppose that $T = CT^*C$ for some conjugation C. Let $\{f_i\}_{i=1}^n$ be a C-real basis, and let U be the matrix whose ith column is f_i. By Proposition 3.3, U is unitary. Also note that if $\{e_i\}$ is the standard basis, then $Ue_i = f_i$. This yields that

$$\begin{aligned} \langle U^*TUe_j, e_i\rangle &= \langle TUe_j, Ue_i\rangle \\ &= \langle Tf_j, f_i\rangle. \end{aligned} \tag{1}$$

Now using Proposition 3.3 and the fact that $C^2 = I$, we can calculate

$$\begin{aligned} \langle Tf_j, f_i\rangle &= \langle CT^*Cf_j, f_i\rangle \\ &= \langle Cf_i, T^*Cf_j\rangle \end{aligned}$$

Since $Cf_i = f_i$, we have

$$\begin{aligned} \langle Cf_i, T^*Cf_j\rangle &= \langle f_i, T^*f_j\rangle \\ &= \langle Tf_i, f_j\rangle, \end{aligned}$$

and thus $\langle Tf_j, f_i\rangle = \langle Tf_i, f_j\rangle$. Substituting from (1) twice yields that

$$\langle U^*TUe_j, e_i\rangle = \langle U^*TUe_i, e_j\rangle. \tag{2}$$

Since for any matrix S the expression $\langle Se_j, e_i\rangle$ gives the (i, j)-th entry of S, we can conclude from (2) that U^*TU is complex symmetric and therefore that T is UECSM.

Conversely, suppose that there is some unitary matrix U such that U^*TU is a complex symmetric matrix. Let C be the conjugation UJU^*, where J denotes the standard conjugation. By Example 3.4, we know that $J(U^*T^*U)J = U^*TU$. Thus

$$\begin{aligned} CT^*C &= (UJU^*)T^*(UJU^*) \\ &= U(JU^*T^*UJ)U^* \\ &= U(U^*TU)U^* \\ &= T. \end{aligned}$$

We conclude that T is C-symmetric, which finishes the proof. □

Lemma 3.5 provides a powerful tool for proving that matrices are UECSM, and we will conclude this section by using it to prove the well-known result that all square matrices are similar to complex symmetric matrices. An alternate proof may be found in [10, Theorem 4.4.9].

Theorem 3.7. *Every square matrix is similar to a complex symmetric matrix.*

Proof. The Jordan Canonical Form Theorem [10, Section 3.1] states that every matrix is similar to a direct sum of Jordan blocks $J_{m_i}(\lambda_i)$, where an $m \times m$ Jordan block is given by,

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & 0 & & & \lambda \end{bmatrix}.$$

It therefore suffices to prove that any direct sum of Jordan blocks is similar to a complex symmetric matrix, but we will actually show that such matrices are UECSM. By Proposition 2.5, direct sums of UECSM's are UECSM, and thus it suffices to prove that any matrix $J_m(\lambda)$ is UECSM. By adding a multiple of the identity matrix (again, by Proposition 2.5), we need only prove that $J_m(0)$ is UECSM.

Let $T = J_m(0)$ and let C be the flip conjugation

$$C \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \overline{x_m} \\ \vdots \\ \overline{x_1} \end{bmatrix}.$$

If $\{e_i\}$ is the standard basis, we can think of T as a "backward shift" on the $\{e_i\}$. That is, $Te_i = e_{i-1}$ for $i > 1$ and $Te_1 = 0$. Similarly, T^* acts as a "forward shift." We will now show that $T = CT^*C$ by showing that the two transformations agree on the standard basis. Suppose first that $i \neq 1$. In this case,

$$\begin{aligned} CT^*Ce_i &= CT^*e_{m-i+1} \\ &= Ce_{m-i+2} \\ &= e_{i-1} \\ &= Te_i. \end{aligned}$$

We also have

$$CT^*Ce_1 = CT^*e_m = 0 = Te_1.$$

We have shown that T and CT^*C agree on a basis, and therefore $CT^*C = T$. By Lemma 3.5, T is UECSM. □

4. Self-adjoint matrices and the Cartesian decomposition

In general, we will not apply Lemma 3.5 to matrices directly. Instead, we will use the *Cartesian decomposition* of the matrix.

Proposition 4.1 (Cartesian decomposition)**.** *Given any $T \in M_n(\mathbb{C})$, there exist unique self-adjoint matrices A and B such that $T = A + iB$. Moreover, A and B are given by*

$$A = \frac{1}{2}(T + T^*), \quad \textit{and} \quad B = \frac{1}{2i}(T - T^*).$$

Proof. It is easy to verify that A and B are self-adjoint, and that $T = A + iB$. We must show that these choices of A and B are unique. If $T = A + iB$ and $T = A' + B'$, then

$$0 = (A - A') + i(B - B').$$

Thus it suffices to show that the zero matrix has a unique Cartesian decomposition. Suppose $0 = A + iB$ where A and B are self-adjoint. Since 0 is self-adjoint, we can calculate

$$\begin{aligned} A + iB &= 0 \\ &= 0^* \\ &= A - iB. \end{aligned}$$

Adding the first and last formula gives $2A = 0$, which in turn gives $B = 0$, as desired. □

The formulas for A and B provided by Proposition 4.1 give us another characterization of C-symmetry.

Proposition 4.2. *A matrix $T = A + iB$ is C-symmetric if and only if both A and B are C-symmetric for the same C.*

Proof. Recall that linear combinations of C-symmetric matrices are C-symmetric, and that adjoints of C-symmetric matrices are C-symmetric. Combining these facts with the formulas from Proposition 4.1, we are done. □

The previous proposition indicates that self-adjoint matrices will play a special role in our investigation. We will now digress briefly to present some of their basic properties.

Definition. A matrix T is called *normal* if $T^*T = TT^*$.

In particular, all self-adjoint matrices are normal. Normal matrices are characterized by the Spectral Theorem [10, Theorem 2.5.4].

Theorem 4.3 (The Spectral Theorem)**.** *A matrix T is normal if and only if it is unitarily equivalent to a diagonal matrix. That is, T is normal if and only if it has an orthonormal basis of eigenvectors.*

A consequence of the Spectral Theorem is that all normal matrices are UECSM, and therefore C-symmetric for some conjugation C. Proposition 4.2 says that in order to determine whether or not T is UECSM, we need to check whether A and B are *simultaneously* C-symmetric for some shared C. To better understand this problem, we will now turn out attention to describing all of the conjugations C with respect to which a given self-adjoint matrix is C-symmetric.

Lemma 4.4. *If A is a C-symmetric self-adjoint matrix, then there exists a C-real orthonormal basis of eigenvectors of A.*

Proof. The first step is to show that all of the eigenvalues of A are real. Throughout the proof, E_λ will denote the eigenspace of A corresponding to the eigenvalue λ. Now suppose that λ is an eigenvalue of A and $x \in E_\lambda \setminus \{0\}$. Then

$$\begin{aligned} \lambda\|x\|^2 &= \lambda\langle x, x\rangle \\ &= \langle \lambda x, x\rangle \\ &= \langle Ax, x\rangle \\ &= \langle x, Ax\rangle \\ &= \langle x, \lambda x\rangle \\ &= \overline{\lambda}\|x\|^2. \end{aligned}$$

Since $x \neq 0$, we can cancel to obtain $\lambda = \overline{\lambda}$ and thus $\lambda \in \mathbb{R}$.

Now suppose that A is C-symmetric. That is, $CAC = A$ and so $CA = AC$. If λ is an eigenvalue of A and $x \in E_\lambda$, then

$$ACx = CAx = C\lambda x = \lambda Cx,$$

where we used that $\lambda \in \mathbb{R}$ in the final step. This calculation shows that Cx is also an eigenvector of A, again with eigenvalue λ. That is, $C(E_\lambda) \subseteq E_\lambda$. Moreover, if $x \in E_\lambda$, then $Cx \in E_\lambda$ and $C(Cx) = x$. This implies that we actually have $C(E_\lambda) = E_\lambda$. Consequently, C is an $\mathbb{R}$-linear bijection of each E_λ onto itself, and it is easy to check that $C \restriction_{E_\lambda}$ is a conjugation of E_λ. If $\{\lambda_1, \ldots, \lambda_m\}$ is the set of distinct eigenvalues of A, then the Spectral Theorem says that

$$\mathbb{C}^n = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m},$$

where $\oplus$ indicates an orthogonal direct sum. We can therefore decompose

$$C = C_1 \oplus \cdots \oplus C_m, \qquad \text{where} \qquad C_i = C \restriction_{E_{\lambda_i}}.$$

By Lemma 3.6, each E_{λ_i} has a C_i-real orthonormal basis, and the orthogonal sum of these bases will be a C-real orthonormal basis for $\mathbb{C}^n$. $\square$

Now that we understand how the C-symmetries of a self-adjoint matrix arise, we can revisit Lemma 4.2 and see what this says about the general case of $T = A + iB$.

Lemma 4.5. *If $T = A + iB$ is an $n \times n$ matrix, then T is unitarily equivalent to a symmetric matrix if and only if there exist orthonormal bases of eigenvectors $\{e_i\}$ and $\{f_i\}$ of A and B, respectively, such that $\langle e_i, f_j \rangle \in \mathbb{R}$ for all $1 \leq i, j \leq n$.*

Proof. First assume that T is UECSM. By Lemmas 3.5 and 4.2, there is a conjugation C on $\mathbb{C}^n$ such that $CAC = A$ and $CBC = B$. By Lemma 4.4, A and B have C-real orthonormal bases of eigenvectors $\{e_i\}$ and $\{f_i\}$, respectively. For these we have,

$$\langle e_i, f_j \rangle = \langle Ce_i, Cf_j \rangle = \langle f_j, e_i \rangle = \overline{\langle e_i, f_j \rangle},$$

and so $\langle e_i, f_j \rangle \in \mathbb{R}$.

Conversely suppose there exist such $\{e_i\}$ and $\{f_i\}$. Then define $C : \mathbb{C}^n \to \mathbb{C}^n$ by

$$Cx = \sum_{i=1}^{n} \overline{\langle x, e_i \rangle} e_i.$$

This map is evidently conjugate-linear, but we need to check that it is an isometric involution. Observe that $Cx = x$ if and only if $\langle x, e_i \rangle \in \mathbb{R}$ for all i. In particular, $Ce_i = e_i$ and $Cf_i = f_i$ for all i. Next, we claim that $C = UJU^*$, where J is the standard conjugation and U is the unitary matrix which has e_i as its ith column. It is an easy exercise to show that $UJU^* e_i = e_i$, and thus C and UJU^* agree on a basis. By conjugate-linearity, we can conclude that $C = UJU^*$, and thus C is a conjugation (as in Example 3.2).

The final step is to show that both A and B are C-symmetric, which will prove that T is UECSM by Lemmas 3.5 and 4.2. Recall that $Ce_i = e_i$ and $Cf_i = f_i$ for all i. Since we know $Ae_i = \lambda_i e_i$, where $\lambda_i \in \mathbb{R}$, we can calculate

$$CACe_i = CAe_i = \lambda_i Ce_i = \lambda_i e_i = Ae_i.$$

Since A and CAC agree on a basis, it follows from linearity that $Ax = CACx$ for all $x \in \mathbb{C}^n$. One can similarly show that $CBC = B$, and thus T is C-symmetric. $\square$

Remark. In the previous proof, we could have let C be complex conjugation with respect to $\{f_i\}$ instead of with respect to $\{e_i\}$.

Corollary 4.6. *Let $T = A + iB$ be an $n \times n$ matrix that is unitarily equivalent to a complex symmetric matrix, and let $\{e_i\}$ and $\{f_i\}$ be a pair of orthonormal bases of eigenvectors of A and B, respectively, as in Lemma 4.5. Then for any A' and B' such that the $\{e_i\}$ are eigenvectors for one and $\{f_i\}$ are eigenvectors for the other, $T' = A' + iB'$ is unitarily equivalent to a complex symmetric matrix.*

The corollary says that given $T = A+iB$, we can adjust the eigenvalues of A and B however we wish. This supports our intuition that eigenvalues (a similarity invariant) are of little use in determining whether or not a matrix is UECSM.

5. A test for $n \times n$ matrices

Definition. We say that a pair of orthogonal bases $\{g_i\}$ and $\{h_i\}$ for $\mathbb{C}^n$ are *proper* if $\langle g_1, h_1 \rangle \in \mathbb{R}$ and $\langle g_i, h_j \rangle = 0 \implies i \neq 1$ and $j \neq 1$. If we let $M = (\langle g_i, h_j \rangle)_{i,j=1}^n$, then this is equivalent to the top-left entry of M being real while the first row and column contain no zeros.

Our main result is the following theorem.

Theorem 5.1. *Let $T = A + iB$ be an $n \times n$ matrix, and let $\{g_i\}$ and $\{h_i\}$ be any proper pair of orthogonal bases of eigenvectors of A and B, respectively. Then T is unitarily equivalent to a complex symmetric matrix if for $2 \leq i, j \leq n$,*

$$\frac{\langle g_i, h_j \rangle}{\langle g_i, h_1 \rangle \langle g_1, h_j \rangle} \in \mathbb{R}. \tag{3}$$

Moreover, if A and B both have n distinct eigenvalues, then (3) *is also necessary for T to be UECSM.*

Proof. Suppose that (3) holds. Define $e_1 = g_1$, $f_1 = h_1$ and otherwise,

$$e_i = \frac{1}{\langle g_i, h_1 \rangle}\, g_i, \quad f_j = \frac{1}{\langle h_j, g_1 \rangle}\, h_j.$$

Once normalized, these bases satisfy Lemma 4.5, showing that T is UECSM.

Conversely, suppose that T is UECSM and that A and B have n distinct eigenvalues. By Lemma 4.5, there are orthonormal bases of eigenvectors $\{e_i\}$ and $\{f_i\}$ of A and B, respectively, such that $\langle e_i, f_j \rangle \in \mathbb{R}$. As the eigenspaces of A and B are one-dimensional, we can reorder these bases so that $g_i = \omega_i e_i$ and $h_j = \zeta_j f_j$ for unimodular $\omega_i, \zeta_j \in \mathbb{C}$. Then for $2 \leq i, j \leq n$,

$$\begin{aligned}
\frac{\langle g_i, h_j \rangle \langle g_1, h_1 \rangle}{\langle g_i, h_1 \rangle \langle g_1, h_j \rangle} &= \frac{\langle \omega_i e_i, \zeta_j f_j \rangle \langle \omega_1 e_1, \zeta_1 f_1 \rangle}{\langle \omega_i e_i, \zeta_1 f_1 \rangle \langle \omega_1 e_1, \zeta_j f_j \rangle} \\
&= \frac{\omega_i \omega_1 \overline{\zeta_j \zeta_1} \langle e_i, f_j \rangle \langle e_1, f_1 \rangle}{\omega_i \omega_1 \overline{\zeta_j \zeta_1} \langle e_i, f_1 \rangle \langle e_1, f_j \rangle} \\
&= \frac{\langle e_i, f_j \rangle \langle e_1, f_1 \rangle}{\langle e_i, f_1 \rangle \langle e_1, f_j \rangle}
\end{aligned}$$

which is real since each $\langle e_i, f_j \rangle \in \mathbb{R}$. The pair $\{g_i\}$ and $\{h_i\}$ being proper ensures that $\langle g_1, h_1 \rangle \in \mathbb{R} \setminus \{0\}$, and so the preceding equation yields that $\{g_i\}$ and $\{h_i\}$ satisfy (3). $\square$

Remark. The condition (3) of Theorem 5.1 can be visualized using matrices. If $\{e_i\}$ and $\{f_i\}$ are a proper pair of orthogonal bases of eigenvectors, we can consider the matrix $M = (m_{i,j}) = (\langle e_i, f_j \rangle)$, which will be unitary if both bases are normalized. Thinking of M with $1 \times (n-1)$ blocking

$$M = \left[\begin{array}{c|c} m_{1,1} & r_1 \\ \hline c_1 & D \end{array}\right],$$

the condition says that each element in the lower-right block D has the same argument as the product of the first element in its row and the first element in its column.

Using the fact that Lemmas 3.5 and 4.5 are constructive, we can unwind the proof of Theorem 5.1 to explicitly demonstrate the unitary equivalence.

Corollary 5.2. *If a matrix T is verified to be UECSM via Theorem 5.1, then it is possible to explicitly find a unitary matrix that makes T a CSM and a conjugation for which T is C-symmetric.*

Proof. Let $\{e_i\}$ and $\{f_i\}$ be the orthonormal bases from the proof of Theorem 5.1 (which can easily be constructed from the hypothesis data), and let U be the unitary matrix that has e_i (or f_i) as its ith column. The proof of Lemma 4.5 says that T is C-symmetric, where $C = UJU^*$ and J is the standard conjugation. The proof of Lemma 3.5 demonstrates that U^*TU will be a CSM. □

While there are already multiple proofs that all 2×2 matrices are UECSM ([7, Example 6],[2, Corollary 3.3],[9, Corollary 1]), we can use Theorem 5.1 to provide another proof.

Corollary 5.3. *Every 2×2 matrix is unitarily equivalent to a complex symmetric matrix.*

Proof. Let $T = A + iB$ be a 2×2 matrix. We may assume without loss of generality that A and B each have 2 distinct eigenvalues, since otherwise T would be UECSM by Proposition 2.5. Consider now the case when A and B share an eigenvector. Since the eigenvectors of A (and B) are orthogonal by the Spectral Theorem, it must hold that A and B have the same eigenspaces. Thus, A and B are *simultaneously* unitarily diagonalizable, and $T = A + iB$ is UECSM (T is, in fact, normal). Now consider the case when A and B do not share an eigenvector. Let $\{e_i\}$ and $\{f_i\}$ be orthonormal bases of eigenvectors of A and B, respectively. Compute the unitary $U = (u_{i,j}) = (\langle e_i, f_j \rangle)$, and observe that since A and B do not share an eigenvector, no entry of U is equal to 0. Multiply e_1 by a unimodular constant so that $\langle e_1, f_1 \rangle \in \mathbb{R}$. Since the columns of U must be orthogonal, $u_{1,1}\overline{u_{1,2}} + u_{2,1}\overline{u_{2,2}} = 0$ which yields

$$\frac{u_{2,2}}{u_{1,2}u_{2,1}} = -\frac{u_{1,1}}{|u_{1,2}|^2}.$$

As $u_{1,1} \in \mathbb{R}$, it follows that $\frac{u_{2,2}}{u_{1,2}u_{2,1}} \in \mathbb{R}$. By Theorem 5.1, T is unitarily equivalent to a complex symmetric matrix. □

To apply Theorem 5.1, one must have already constructed a proper pair of orthonormal bases. It turns out that this requirement is not particularly onerous. One may always calculate the eigenvectors of self-adjoint matrices, and generically these will be a proper pair. We will examine these issues more in Section 7.

6. Special results for 3 × 3 matrices

In this section we introduce an algorithm for determining whether or not a given 3×3 matrix is unitarily equivalent to a complex symmetric matrix. We first require a few preparatory results. The following proposition allows us to easily answer affirmatively in certain cases.

Proposition 6.1. *Let T be a 3×3 matrix with Cartesian decomposition $A + iB$. If either of the following conditions hold, then T is unitarily equivalent to a complex symmetric matrix:*

(i) A or B has a repeated eigenvalue,

(ii) A and B share an eigenvector.

We must digress briefly before we can prove Proposition 6.1. There is a refinement of the Spectral Theorem for *star-cyclic* normal operators, and what follows is only a special case of the result. For a more general discussion, one can consult [3, Section 2.11].

Definition. If $T \in M_n(\mathbb{C})$ and there is a vector $v \in \mathbb{C}^n$ such that

$$\mathbb{C}^n = \operatorname{span}\{v, Tv, T^2v, \ldots, T^{n-1}v\},$$

then v is called a *cyclic vector for T* and T is called *cyclic.*

Theorem 6.2 (Spectral Theorem for cyclic self-adjoint matrices)**.** *If T is a self-adjoint matrix with cyclic vector v, then there is a unitary matrix U such that UTU^* is diagonal and $Uv = (1, \ldots, 1)$.*

We now proceed to the proof of Proposition 6.1.

Proof of Proposition 6.1. First (i). Note that T is UECSM if and only if $T - \lambda I$ is. If B has a repeated eigenvalue λ, then $B - \lambda I$ either has rank 0 or 1. If $B - \lambda I = 0$, then $T - i\lambda I = A$, which is self-adjoint and therefore UECSM by the Spectral Theorem. If $B - \lambda I$ has rank 1, we can show that T is UECSM by simplifying an earlier result [9, Corollary 5] for the finite dimensional case. Let v be a unit vector that spans $\operatorname{Ran}(B - \lambda I)$, and let

$$\mathcal{M} = \operatorname{span}\{v, Av, A^2 v\}$$

be the cyclic subspace generated by v under A. Clearly $\mathcal{M}$ is invariant under A, and it is easy to check that $\mathcal{M}^\perp$ is as well. Since $A \restriction_{\mathcal{M}^\perp}$ is self-adjoint, and $B \restriction_{\mathcal{M}^\perp} = 0$, we have that $T \restriction_{\mathcal{M}_\perp}$ is UECSM. The matrix $A \restriction_{\mathcal{M}}$ is cyclic self-adjoint, so by the Spectral Theorem we can assume without loss of generality (after conjugating T by a unitary) that $v = (1, \ldots, 1)$ and that $A \restriction_{\mathcal{M}}$ is diagonal. Here, the length of v is the dimension of $\mathcal{M}$, as this is the size of the matrix $A \restriction_{\mathcal{M}}$. Since $\operatorname{Ran}(B - \lambda I) = \operatorname{span}\{v\}$ and $B \restriction_{\mathcal{M}}$ is self-adjoint, we have

$$B \restriction_{\mathcal{M}} = b \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

for some $b \in \mathbb{R}$. Thus $T \restriction_{\mathcal{M}} = A \restriction_{\mathcal{M}} + iB \restriction_{\mathcal{M}}$ is UECSM. This yields that $T = T \restriction_{\mathcal{M}} \oplus T \restriction_{\mathcal{M}^\perp}$ is UECSM. The proof is similar if A has a repeated eigenvalue.

Now (ii). If A and B share an eigenvector, then up to unitary equivalence we know that $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where A_1 and B_1 are 1×1 matrices and A_2 and B_2 are 2×2 matrices. This yields that $T = (A_1 + iB_1) \oplus (A_2 + iB_2)$, and because all 1×1 and 2×2 matrices are UECSM, T is UECSM. □

The next lemma tells us that when Proposition 6.1 does not imply that $T = A + iB$ is UECSM, it is easy to construct a proper pair of orthogonal bases to which one can apply Theorem 5.1.

Lemma 6.3. *If $T = A + iB$ is a 3×3 matrix which does not satisfy either hypothesis of Proposition 6.1, then any two orthogonal bases of eigenvectors $\{e_i\}$ and $\{f_i\}$ of A and B, respectively, can be made proper by reordering them and scaling e_1.*

Proof. Without loss of generality we may assume that $\|e_i\| = \|f_i\| = 1$, since the conditions of being proper are not affected by multiplying the basis vectors by real scalars. Note that no e_i is orthogonal to more than one f_j, since otherwise it would be a scalar multiple of the third element of $\{f_j\}$, a contradiction. Similarly no fixed f_i is orthogonal to more than one e_i. In terms of the unitary matrix $U = (u_{i,j}) = (\langle e_i, f_j \rangle)$, this means that no row or column has more than one 0.

We claim that there is at most one 0 in U. If U had more than one 0, they must be in different rows and columns, so we could reorder the bases so that

$$U = \begin{bmatrix} 0 & * & * \\ * & 0 & * \\ a & b & * \end{bmatrix},$$

where the $*$'s represent arbitrary complex numbers. To preserve the orthogonality of columns, we must have $a = 0$ or $b = 0$. In either case, there must be more than one 0 in a single column, which we have already excluded. By reordering the bases, we can ensure that the 0 entry is not in the first row or column. If $\langle e_1, f_1 \rangle \notin \mathbb{R}$, multiply e_1 by $\frac{|\langle e_1, f_1 \rangle|}{\langle e_1, f_1 \rangle}$ and then $\{e_i\}$ and $\{f_i\}$ will be a proper pair. □

Using the preceding results, we can construct an algorithm that will decide whether or not a 3×3 matrix T is unitarily equivalent to a complex symmetric matrix. Since none of the operations are more complicated than finding roots of cubic polynomials, it can be performed using exact values, assuming the data is given exactly. We implemented it in `Mathematica` without much difficulty, and the code can be found in Appendix A. The algorithm is:

Algorithm. Given a 3×3 matrix T,

(1) Compute $A = \frac{1}{2}(T + T^*)$ and $B = \frac{1}{2i}(T - T^*)$.

(2) Compute the eigenvalues of A and B. If either A or B has a repeated eigenvalue, then T is UECSM by Proposition 6.1.

(3) Compute arbitrary sets of eigenvectors $\{g_i\}$ and $\{h_i\}$ of A and B, respectively, and compute the matrix

$$M = (m_{i,j}) = (\langle g_i, h_j \rangle)_{i,j}.$$

(4) If M has more than one entry equal to 0, then T is UECSM (Lemma 6.3, Proposition 6.1). Otherwise, reorder the rows and columns of M so that the 0 entry is not in the first row or column (so that $\{g_i\}$ and $\{h_i\}$ form a proper pair). Scale g_1 by $\frac{|\langle g_1, h_1 \rangle|}{\langle g_1, h_1 \rangle}$.

(5) By Theorem 5.1, T is UECSM if and only if for all $2 \leq i, j \leq 3$,

$$\frac{m_{i,j}}{m_{1,j} m_{i,1}} \in \mathbb{R}.$$

(6) If T is UECSM, one can exhibit a corresponding conjugation C by first normalizing $\{g_i\}$ and scaling g_2 and g_3 so that $\langle g_i, h_1 \rangle \in \mathbb{R}$ for all i. If $U = \left[\begin{array}{c|c|c} g_1 & g_2 & g_3 \end{array} \right]$, then by the proof of Corollary 5.2 we have that U^*TU is complex symmetric and T is C-symmetric with respect to the conjugation $C = UJU^*$ (where J is the standard conjugation).

It is worth noting that steps 1, 3 and 5 carry through to the $n \times n$ case as long as A and B have n distinct eigenvalues and a proper pair of bases can be found. Step 4 is no longer valid, as for $n > 3$ the preceding conditions do not guarantee that T is UECSM. A generalization of this algorithm to $n \times n$ matrices will be discussed in Section 7. The following examples illustrate the steps of the algorithm.

Example 6.4. Theorem 5.1 provides another proof of the fact from [9, Example 1] that for $a, b \neq 0$ the matrix

$$T = \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}$$

is UECSM if and only if $|a| = |b|$. By dividing by b, it is enough to consider matrices of the form

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix}.$$

For this T, one can verify that

$$A = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{a}{2} \\ 0 & \frac{\overline{a}}{2} & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & -\frac{ia}{2} \\ 0 & \frac{i\overline{a}}{2} & 0 \end{bmatrix}.$$

Furthermore, the eigenvalues of both A and B are

$$\{0, \frac{1}{2}\sqrt{1+|a|^2}, -\frac{1}{2}\sqrt{1+|a|^2}\},$$

eigenvectors of A are

$$g_1 = \begin{bmatrix} a \\ 0 \\ -1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ \sqrt{1+|a|^2} \\ \overline{a} \end{bmatrix}, \quad g_3 = \begin{bmatrix} 1 \\ -\sqrt{1+|a|^2} \\ \overline{a} \end{bmatrix},$$

and eigenvectors of B are

$$h_1 = \begin{bmatrix} a \\ 0 \\ 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 1 \\ i\sqrt{1+|a|^2} \\ -\overline{a} \end{bmatrix}, \quad h_3 = \begin{bmatrix} 1 \\ -i\sqrt{1+|a|^2} \\ -\overline{a} \end{bmatrix}.$$

The matrix $M = (\langle g_i, h_j \rangle)_{i,j=1}^3$ required by the algorithm is given by:

$$M = \begin{bmatrix} 1-|a|^2 & 2 & 2 \\ 2 & \beta & \overline{\beta} \\ 2 & \overline{\beta} & \beta \end{bmatrix}.$$

where $\beta = 1+i-\frac{1-i}{|a|^2}$. If $|a| \neq 1$, then the bases $\{g_i\}$ and $\{h_i\}$ are proper and since β has non-zero imaginary component, T is not UECSM by Theorem 5.1.

If $|a| = 1$, then $\beta = 2i$. We can relabel the vectors of each basis and scale the new e_1 by $-i$ to get the matrix,

$$M' = 2 \begin{bmatrix} 1 & -1 & -i \\ -i & i & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

It is easy to check that $\frac{m_{i,j}}{m_{i,1}m_{1,j}} \in \mathbb{R}$ for $i, j \geq 2$, so T is UECSM by Theorem 5.1.

Example 6.5. Let

$$T = \begin{bmatrix} 1+4i & (-2-i)\sqrt{2} & -1-4i \\ i\sqrt{2} & 0 & i\sqrt{2} \\ -1 & (2-i)\sqrt{2} & 1 \end{bmatrix}.$$

In this example, we prove that $T = A + iB$ is unitarily equivalent to a complex symmetric matrix and use the full algorithm to find a conjugation C with respect to which T is C-symmetric. Per step 2, we first calculate the eigenvalues of A and B, which are

$$\left\{2\left(1+\sqrt{2}\right), -2, 2\left(1-\sqrt{2}\right)\right\} \quad \text{and} \quad \left\{2\left(1+\sqrt{3}\right), 2\left(1-\sqrt{3}\right), 0\right\},$$

respectively. Neither A nor B has a repeated eigenvalue, and so we calculate eigenvectors of A:

$$g_1 = \begin{bmatrix} -1-2i\sqrt{2} \\ 2+i\sqrt{2} \\ 3 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 \\ -i\sqrt{2} \\ 1 \end{bmatrix}, \quad g_3 = \begin{bmatrix} -1+2i\sqrt{2} \\ -2+i\sqrt{2} \\ 3 \end{bmatrix},$$

and eigenvectors of B:

$$h_1 = \begin{bmatrix} -1-\frac{2}{\sqrt{3}} \\ i\sqrt{\frac{2}{3}} \\ 1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} -1+\frac{2}{\sqrt{3}} \\ -i\sqrt{\frac{2}{3}} \\ 1 \end{bmatrix}, \quad h_3 = \begin{bmatrix} 1 \\ i\sqrt{2} \\ 1 \end{bmatrix}.$$

Since A and B do not share an eigenvector, we must use the full algorithm and not a shortcut provided by Proposition 6.1. Next we compute $M = (\langle g_i, h_j \rangle) =$

$$\begin{bmatrix} \frac{2}{3}\left(2+i\sqrt{2}\right)\left(3+\sqrt{3}\right) & -\frac{2}{3}i\left(-2i+\sqrt{2}\right)\left(-3+\sqrt{3}\right) & 4-4i\sqrt{2} \\ -\frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & 0 \\ \frac{2}{3}\left(2-i\sqrt{2}\right)\left(3+\sqrt{3}\right) & \frac{2}{3}i\left(2i+\sqrt{2}\right)\left(-3+\sqrt{3}\right) & 4+4i\sqrt{2} \end{bmatrix}.$$

If we let $\alpha = \frac{|m_{1,1}|}{m_{1,1}}$ and $\beta = \frac{|m_{3,1}|}{m_{3,1}}$, rewriting M with respect to $\{\alpha g_1, g_2, \beta g_3\}$ and $\{h_1, h_2, -ih_3\}$ we get,

$$M' = \begin{bmatrix} 2\left(\sqrt{2}+\sqrt{6}\right) & 2\sqrt{2}\left(-1+\sqrt{3}\right) & 4\sqrt{3} \\ -\frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} & 0 \\ 2\left(\sqrt{2}+\sqrt{6}\right) & 2\sqrt{2}\left(-1+\sqrt{3}\right) & -4\sqrt{3} \end{bmatrix}.$$

As all of the entries are real, Theorem 1 says that T is UECSM. Letting $\{e_i\} = \{\frac{\alpha g_1}{\|\alpha g_1\|}, \frac{g_2}{\|g_2\|}, \frac{\beta g_3}{\|\beta g_3\|}\}$ and $U = [e_1|e_2|e_3]$, we can construct a conjugation C for which T is C-symmetric,

$$Cx = UU^t\overline{x} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{\sqrt{2}} & -\frac{1}{2} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{i}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix},$$

and a complex symmetric matrix that T is unitarily equivalent to,

$$U^*TU = 2\begin{bmatrix} 1+\sqrt{2}+i & -i & i \\ -i & -1 & -i \\ i & -i & 1-\sqrt{2}+i \end{bmatrix}.$$

Example 6.6. Consider the following matrices, which are all similar to each other:

$$T_1 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 6 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{bmatrix}.$$

Applying the same method as in the previous example yields that T_1 is C-symmetric where

$$C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = V \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ \overline{x_3} \end{bmatrix},$$

and

$$V = \begin{bmatrix} \frac{6\left(-19+6i\sqrt{74}\right)}{3025} & \frac{42\left(19-6i\sqrt{74}\right)}{3025} & \frac{7}{605}\left(19-6i\sqrt{74}\right) \\ \frac{42\left(19-6i\sqrt{74}\right)}{3025} & \frac{19\left(-19+6i\sqrt{74}\right)}{3025} & \frac{6}{605}\left(19-6i\sqrt{74}\right) \\ \frac{7}{605}\left(19-6i\sqrt{74}\right) & \frac{6}{605}\left(19-6i\sqrt{74}\right) & \frac{6}{605}\left(-19+6i\sqrt{74}\right) \end{bmatrix}.$$

We also get that T_1 is unitarily equivalent to

$$T_1' = \begin{bmatrix} \frac{56}{37} - i\sqrt{\frac{37}{2}} & -\frac{55}{37} & \frac{35\sqrt{55}}{74} \\ -\frac{55}{37} & \frac{56}{37} + i\sqrt{\frac{37}{2}} & \frac{35\sqrt{55}}{74} \\ \frac{35\sqrt{55}}{74} & \frac{35\sqrt{55}}{74} & \frac{147}{37} \end{bmatrix}.$$

We also get that T_2 and T_3 are not UECSM. It is easy to check that if T_2 has Cartesian decomposition $T_2 = A + iB$, both A and B have 3 distinct eigenvalues and that they do not share an eigenvector. None of the eigenspaces of A are orthogonal to any of the eigenspaces of B, and so it is easy to construct a proper pair of orthogonal bases of eigenvectors of A and B. It is also easy to check that these will not satisfy condition (3) of Theorem 5.1. Similarly for T_3.

7. Applications for generic $n \times n$ matrices

One scenario in which it is useful to have a computerized test for a matrix being UECSM is in answering questions of the form "Are all matrices with property P UECSM?" One can generate random matrices with property P, and test them for being UECSM. However, for an arbitrary $n \times n$ matrix, Theorem 5.1 only provides a sufficient condition for being unitarily equivalent to a complex symmetric matrix, and it presupposes having found a proper pair of orthonormal bases for A and B. The next result show that in practice, Theorem 5.1 will be sufficient and a proper pair of orthonormal bases can easily be constructed as in Lemma 6.3.

Proposition 7.1.

(i) The set of matrices that have a repeated eigenvalue has measure zero and is nowhere dense.

(ii) The set of $T = A + iB$ where either A or B has a repeated eigenvalue has measure zero and is nowhere dense.

(iii) The set of $T = A + iB$ where A and B share an eigenvector has measure zero and is nowhere dense.

Proof sketch. Assertion (i) is a standard result, and a treatment can be found in [4, Proposition 5.11]. Assertion (ii) follows easily from (i). We will only outline a proof of (iii). If $T = A + iB$ and A and B share an eigenvector, we can find a matrix $T' = A' + iB$ that is arbitrarily close to T such that A' and B do not share an eigenvector. This can be accomplished by replacing A with U^*AU, where U is an appropriate unitary satisfying $\|U - I\| < \epsilon$. This shows that the set in question has empty interior. Since it is evidently closed, we have proved the second assertion of (iii). The first assertion can be obtained via a dimension counting argument. Let $\mathcal{A}_n$ denote the set of self-adjoint matrices with n distinct eigenvalues. By (ii), we only need to consider $T = A + iB$ where $A, B \in \mathcal{A}_n$. We can identify the set of such T with the Cartesian product $\mathcal{A}_n \times \mathcal{A}_n$. Let $\mathcal{U}_n$ be the set of $n \times n$ unitary matrices, modulo the equivalence relation $U_1 \sim U_2$ if and only if there exists a diagonal matrix D with unimodular diagonal entries such that $U_1 = DU_2$. Also, let

$$\mathcal{R}_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_1 > \cdots > \lambda_n\}.$$

By the Spectral Theorem, we can identify $\mathcal{A}_n$ with $\mathcal{U}_n \times \mathcal{R}_n$. It is easy to check that $\mathcal{A}_n$ is described by n^2 real parameters, and that $\mathcal{R}_n$ is described by n real parameters, and so $\mathcal{U}_n$ has $n(n-1)$ real parameters. The set of $T = A + iB$ where A and B both have simple spectra can be identified with

$$\mathcal{A}_n^2 \cong \mathcal{R}_n^2 \times \mathcal{U}_n^2.$$

By similar reasoning, the set of $T = A+iB$ where A and B have simple spectra but share an eigenvector is a finite union of sets of the form

$$\mathcal{R}_n^2 \times \mathcal{U}_n \times \mathcal{U}_k, \tag{4}$$

ranging over all $k < n$. By the formula obtained above, if $k < n$, then $\mathcal{U}_k$ requires fewer parameters than $\mathcal{U}_n$. Thus any set of the form (4) has real dimension strictly smaller than $2n^2$, and therefore has measure zero. □

This proposition says that the algorithm from Section 5 applies to all $n \times n$ matrices outside of a nowhere dense set of measure zero.

We will conclude with a discussion of how we applied our algorithm to the question "Are all rank-two 4×4 partial isometries UECSM?" First, some standard terminology.

Definition. The *support* of a matrix T is the set $(\ker T)^{\perp}$. A matrix T is a *partial isometry* if $\|Tx\| = \|x\|$ for all x in the support of T.

The first thing we did was to simplify the search space using unitary equivalence.

Proposition 7.2. *Each rank-two 4×4 rank-two partial isometry is unitarily equivalent to a matrix of the form UP, where U is unitary and*

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This proposition is simple to prove using the *Polar Decomposition* of a matrix [10, Section 7.3], which we will not discuss here. It says that to generate random rank-two partial isometries, it suffices to generate random unitary matrices. The following remarks provide a way to do this.

Definition. If $T \in M_n(\mathbb{C})$, let e^T denote the *matrix exponential* of T, given by

$$e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

Remark. It is an easy calculus exercise to show that the above series converges absolutely for every matrix T. Since $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ is complete, it follows that e^T is well-defined.

Proposition 7.3.

(i) $(e^T)^* = e^{T^*}$

(ii) If $ST = TS$, *then* $e^S e^T = e^{S+T}$.

(iii) If $A = A^*$, *then* e^{iA} *is unitary.*

Proof. Since T^* is just the conjugate-transpose of T, it is easy to see that $T_n \to T$ if and only if $T_n^* \to T^*$. A simple computation using this fact and the additivity of the adjoint operation proves (i). The proof of (ii) is just a straightforward application of the Binomial Theorem, and (iii) follows directly from (i), (ii) and the observation that $e^0 = I$. □

One can therefore generate unitary matrices by generating self-adjoint matrices A (with entries uniformly distributed in, say, the set $\{x + iy : |x|, |y| < 1\}$) and taking the matrix exponential of iA.

We used `Mathematica` to test $100,000$ matrices of the form UP, with random unitary components generated as described above. All of them were UECSM, and so we conjectured that every rank-two 4×4 partial isometry is unitarily equivalent to a complex symmetric matrix. This fact was soon confirmed.

8. Acknowledgments

We wish to thank Stephan Garcia, without whom this thesis would not have been possible.

Appendix A. Algorithm implementation

```
(* ********************
UECSM Test, Version 0.90
2/19/2008
Author- James Tener

This code contains the routine UECSMTest, which will test whether or
not an inputted matrix T is UECSM.  If T is 3x3, the test will always
return an answer, but for other matrices there is no guarantee.
Under most circumstances, if the matrix is UECSM the code will output
a unitary matrix U and the complex symmetric matrix obtained from T
by conjugating by U.

******************** *)

(* Parameters *)
(* Error is the tolerance used in checking approximate equality
Exact is a boolean that indicates whether the program attempts to do
exact calculations *)
Exact = False;
Error = 0.0000001;
DisplayMessages = True;

(* The UECSM test return True if T is UECSM, False if it is not, and
-1 if it cannot tell..
If T is 3x3, UECSMTest will call SmallUECSMTest, which will give an
answer for any 3x3 matrix.
If Loud is set to false, message output will be suppressed *)
UECSMTest[T_, Loud_: True] :=
  Module[{T2, A, B, i, j, l, m, e, f,
    U = Array[Ze, {Length[T], Length[T]}], properU, propere, properf,
    size = Length[T]},
   If[Not[Length[T] == Length[Transpose[T]]],
    If[Loud,
     Print["Error: UECSMTest only applies to square matrices"];];
    Return[-1];];
```

```
 (* Use SmallUECSMTest if possible *)
 If[size == 3, Return[SmallUECSMTest[T, Loud]];];
 If[size < 3, Return[True];];
 If[Not[Exact], T2 = T + 0., T2 = T];
 A = 1/2 (T2 + ConjugateTranspose[T2]);
 B = 1/2/Sqrt[-1] (T2 - ConjugateTranspose[T2]);
 e = Eigenvectors[A];
 f = Eigenvectors[B];

 (* Construct inner product matrix U (not normalized to be unitary)*)


 For[i = 1, i <= size, i++,
  For[j = 1, j <= size, j++,
   U[[i, j]] = e[[i]].Conjugate[f[[j]]];]];

 (* Make U proper.
    We also pass in e and f so that the changes in U are reflected in
the bases *)
 {properU, propere, properf} = MakeProper[U, e, f];
 If[Not[CheckProper[properU]],
  If[Loud,
   Print["UECSMTest was unable to find a proper pair of bases for A
and B.  Cannot determine if T is UECSM."];];
  Return[-1];];

 (* Run the test *)
 For[i = 2, i <= size, i++,
  For[j = 2, j <= size, j++,
    (* If any of the terms do not line up properly,
    return False indicating not a CSO *)
    If[Not[
       Eq[Im[properU[[i, j]]/properU[[1, j]]/properU[[i, 1]]], 0]],
      If[Loud, Print["The matrix is not UECSM."];];
       Return[False];];
    ];
  ];

 (* At this point, T is UECSM,
```

```
   so exhibit the unitary equivalence *)
   (* First normalize the bases *)
   For[i = 1, i <= size, i++,
    propere[[i]] = propere[[i]]/Norm[propere[[i]]];
    properf[[i]] = properf[[i]]/Norm[properf[[i]]];
    ];
   (* Now fix the bases to make top row and column of U real *)
   For[i = 2, i <= size, i++,
    properf[[i]] = properf[[i]]*Abs[U[[1, i]]]/Conjugate[U[[1, i]]];
    propere[[i]] = propere[[i]]*Abs[U[[i, 1]]]/U[[i, 1]];
    ];
   (* either e or f will make T UECSM.  I choose e. *)
   V = Transpose[propere];

   If[Loud,
    Print["U = ", MF[V]];
    Print["U*TU = ", MF[CT[V].T2.V]];];

   Return[True]];

(* This version of the UECSM test only applies to 3x3 matrices.  It
should give a True or False answer for any such matrix, and if the
matrix is UECSM, and it is non-degenerate, then it will exhibit the
unitary equivalence *)
SmallUECSMTest[T_, Loud_: True] :=
  Module[{T2, A, B, i, j, l, m, e, f, U = Array[Ze, {3, 3}], properU,
    propere, properf},
   If[Not[And[Length[T] == 3, Length[Transpose[T]] == 3]],
    If[Loud,
     Print["Error: SmallUECSMTest only supports 3x3 matrices"];];
    Return[-1];];

   If[Not[Exact], T2 = T + 0., T2 = T];
   A = 1/2 (T2 + ConjugateTranspose[T2]);
   B = 1/2/Sqrt[-1] (T2 - ConjugateTranspose[T2]);
   l = Eigenvalues[A];
   m = Eigenvalues[B];
   (* Test for repeated eigenvalue *)
   If[Or[RepeatedElt[l], RepeatedElt[m]],
```

```
    If[Loud, Print["CSO due to repeated eigenvalue in A or B."];
      Return[True];];];

   e = Eigenvectors[A];
   f = Eigenvectors[B];

   (* Construct inner product matrix U (not normalized to be unitary)*)


   For[i = 1, i <= 3, i++,
    For[j = 1, j <= 3, j++,
     U[[i, j]] = e[[i]].Conjugate[f[[j]]];]];

   (* Check for shared eigenvectors between A and B *)
   If[Not[CheckZerosSmall[U]],
    If[Loud,
      Print["CSO due to shared eigenvector between A and B"];
      Return[True];];];

   (* Make U proper.
   We also pass in e and f so that the changes in U are reflected in
the bases *)
   {properU, propere, properf} = MakeProper[U, e, f];
   If[Not[CheckProper[properU]],
    If[Loud,
      Print[
       "Error: MakeProper failed to make non-degenerate 3x3 proper.
This should not occur."];
      Return[-1];];];

   (* Run the test *)
   For[i = 2, i <= 3, i++,
    For[j = 2, j <= 3, j++,
      (* If any of the terms do not line up properly,
      return False indicating not a CSO *)
      If[Not[
         Eq[Im[properU[[i, j]]/properU[[1, j]]/properU[[i, 1]]], 0]],
        Return[False];];
      ];
```

```
    ];

   (* At this point, T is UECSM,
   so exhibit the unitary equivalence *)
   (* First normalize the bases *)
   For[i = 1, i <= 3, i++,
    propere[[i]] = propere[[i]]/Norm[propere[[i]]];
    properf[[i]] = properf[[i]]/Norm[properf[[i]]];
    ];
   (* Now fix the bases to make top row and column of U real *)
   For[i = 2, i <= 3, i++,
    properf[[i]] = properf[[i]]*Abs[U[[1, i]]]/Conjugate[U[[1, i]]];
    propere[[i]] = propere[[i]]*Abs[U[[i, 1]]]/U[[i, 1]];
    ];
   (* either e or f will make T UECSM.  I choose e. *)
   V = Transpose[propere];

   If[Loud,
    Print["U = ", MF[V]];
    Print["U*TU = ", MF[CT[V].T2.V]];];

   Return[True]
   ];

(********** Subroutines *************)

(* This routine seeks to make U proper by moving 0's out of the first
row and out of the first column, as well as making the top-left entry
real.  It is guaranteed to work if U is 3x3 and CheckZeros returns
True.  Otherwise, there is no guarantee. *)
MakeProper[U_, e_, f_] :=
  Module[{newU = U, i, j, size = Length[U], goodrow = 1, goodcol = 1,
    foundzero = False, newe = e, newf = f, omega},
   (* Find a row without a 0 *)
   For[i = 1, i <= size, i++,
    foundzero = False;
    For[j = 1, j <= size, j++,
     If[Eq[U[[i, j]], 0], foundzero = True;];
     ];
```

```
    If[Not[foundzero], goodrow = i; Break[];];
    ];

   (* Find a column without a 0 *)
   For[i = 1, i <= size, i++,
    foundzero = False;
    For[j = 1, j <= size, j++,
     If[Eq[U[[j, i]], 0], foundzero = True;];
     ];
    If[Not[foundzero], goodcol = i; Break[];];
    ];

   (* Move our good row/column to row 1 and col 1 *)
   newU = SwapRows[newU, 1, goodrow];
   newe = SwapRows[newe, 1, goodrow];
   newU = SwapCols[newU, 1, goodcol];
   newf = SwapRows[newf, 1, goodcol];

   (* Now make the top-left entry real by scaling the first row*)
   If[Not[Eq[Im[U[[1, 1]]], 0]],
    omega = Abs[newU[[1, 1]]]/newU[[1, 1]];
    newU[[1]] = newU[[1]]*omega;
    newe[[1]] = newe[[1]]*omega;
    ];

   Return[{newU, newe, newf}];
   ];

(* Returns true iff the matrix U is proper *)
CheckProper[U_] := Module[{i, size = Length[U]},
   (* Make sure U[[1,1]] is real *)
   If[Not[Eq[Im[U[[1, 1]]], 0]], Return[False];];
   (* Make sure the first row/col contain no zeros *)
   For[i = 1, i <= size, i++,
    If[Eq[U[[1, i]], 0], Return[False];];
    If[Eq[U[[i, 1]], 0], Return[False];];
    ];
   Return[True];
   ];
```

```
(* CheckZeros takes a matrix U, and returns True iff U has at most
one zero in each row, and at most one zero in each column.  Only used
for 3x3 matrices, to test if there is a shared eigenvector between A
and B. *)
CheckZerosSmall[U_] :=
  Module[{i, j, Ut = Transpose[U], zerocount = 0, size = 3},
   (* Check rows *)
   For[i = 1, i <= size, i++,
    For[j = 1, j <= size, j++,
     If[Eq[U[[i, j]], 0], zerocount++];
     If[zerocount > 1, Return[False];];
     ];
    zerocount = 0;
    ];

   (* Check columns *)
   For[i = 1, i <= size, i++,
    For[j = 1, j <= size, j++,
     If[Eq[U[[j, i]], 0], zerocount++];
     If[zerocount > 1, Return[False];];
     ];
    zerocount = 0;
    ];
   Return[True];];

(****** UTILITIES ******)

(* Function to test for equality.  Tests for exact equality of
Exact==True, otherwise returns true iff the absolute difference is
less than Error  *)
Eq[x_, y_] :=
  Module[{rtn},
   If[Exact, rtn = (x == y), rtn = (Abs[x - y] < Error)]; rtn];

(* Returns true iff list contains a repeated element *)
RepeatedElt[list_] := Module[{i, j},
   For[i = 1, i <= Length[list], i++,
    For[j = i + 1, j <= Length[list], j++,
```

```
      If[Eq[list[[i]], list[[j]]], Return[True];];
      ];
    ];
   Return[False];];

(* Returns a new matrix obtained from U by swapping the rows (cols)
r1 and r2  (c1 and c2) *)
SwapRows[U_, r1_, r2_] := Module[{newU = U, temp},
   temp = newU[[r1]];
   newU[[r1]] = U[[r2]];
   newU[[r2]] = temp;
   newU];
SwapCols[U_, c1_, c2_] :=
  Module[{newU = U},
   Return[Transpose[SwapRows[Transpose[newU], c1, c2]]];];

(* Some abbreviations of commonly used functions *)
Ze[x_, y_] := 0;
MF = MatrixForm;
CT = ConjugateTranspose;
```

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